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Smoothness and Rotundity in Quotient Spaces[1]

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This note is devoted to two questions of G. Köthe and M. M. Day: To what extent is smoothness or rotundity of a normed linear space inherited by its quotient spaces? Can smoothness or rotundity of a normed linear space be characterized by that of certain of its quotient spaces? By relating these questions to the problem of existence of best approximations, the only known positive answer to the first question is generalized and the second question is answered in the affirmative. As a consequence, it is obtained that the existence of best approximations is not an invariant under equivalent smooth renorming. Finally, the possibility of a "reasonable" complete answer to the first question is discussed.

Let X be a real or complex normed linear space. The *unit ball* of X is the set $\{x \in X: \|x\| \leqslant 1\}$. The *unit sphere* of X is the boundary of the unit ball of X, i.e., the set $\{x \in X: \|x\| = 1\}$. X is called *smooth*, resp. *rotund*, if every point of the unit sphere of X is a smooth point, resp. an extreme point, of the unit ball of X. (Recall that an element x of the unit sphere of X is called a *smooth point* of the unit ball of X if there exists exactly one element x^* of the unit sphere of the dual space X^* of X such that $x^*x = 1$[2] and, further, that an element x of the unit sphere of X is called an *extreme point* of the unit ball of X if $x = \frac{1}{2}(x_1 + x_2)$ with x_1, x_2 in the unit sphere of X implies that $x = x_1 = x_2$.)

Since it is easily verified that a complex normed linear space is smooth, resp. rotund, if and only if it has the corresponding property when regarded as a real normed linear space, we assume henceforth all normed linear spaces to be real.

If A is a closed linear subspace of a normed linear space X, π denotes the canonical projection of X onto the quotient space X/A which is equipped with the canonical norm $\|\pi x\| := \inf\{\|x - a\| : a \in A\} =: \operatorname{dist}(x, A)$. The adjoint π^* of π, which maps an element f of $(X/A)^*$ onto the composition $f \circ \pi$ in X^*, is then an isometric isomorphism of $(X/A)^*$ onto the annihilator $A^\perp$ of A in X^*.

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[1] Presented to the American Mathematical Society, January 27, 1969.

[2] That at least one such element x^* exists is a consequence of the Hahn–Banach separation theorem.

A is called an *existence subspace* of X if for every element x of X there is a *best approximation* in A, i.e., an element a of A such that $\|x - a\| = \|\pi x\|$. We remark that the property of having best approximations in A is compatible with the equivalence relation defined by π, i.e., for every element x of X either all elements of $\pi^{-1}(\{\pi x\})$ have best approximations in A or none has.

It was shown by V. Klee ([3]; Prop. 3.3) that smoothness of a normed linear space need not be inherited by all of its quotient spaces, by proving that, given a separable normed linear space X and a nonreflexive closed linear subspace A of X with codimension at least two, X can be equivalently renormed in such a way that it becomes smooth, while X/A does not. Earlier, M. M. Day ([2]; p. 114) had shown the same for rotundity by proving that for each infinite set I, there exists a rotund isomorph of the real Banach space $l_1(I)$ that has non-rotund quotient spaces.

In the following lemma we show—using the language of the theory of best approximations—that smoothness or rotundity of a normed linear space is not completely lost under the formation of quotient spaces. As a corollary we obtain necessary and sufficient conditions for the properties under question to be transmitted to a quotient space.

LEMMA. *Let X be a normed linear space and A a closed linear subspace of X.*

(i) *Smoothness of X implies that every element πx of the unit sphere of the quotient space X/A with the property that x has a best approximation in A, is a smooth point of the unit ball of X/A.*

(ii) *Rotundity of X implies that no convex subset of the unit sphere of the quotient space X/A contains more than one element πx with the property that x has a best approximation in A.*

Proof. For the proof of (i) we assume that X is smooth. Let πx be in the unit sphere of X/A and be such that x has a best approximation a in A. If f_1 and f_2 in the unit sphere of the dual $(X/A)^*$ of X/A are such that $f_1(\pi x) = f_2(\pi x) = 1$, $\pi^* f_1$ and $\pi^* f_2$ are in the unit sphere of $A^\perp$ and $\pi^* f_1(x - a) = \pi^* f_2(x - a) = 1$. Since X is smooth, this implies $\pi^* f_1 = \pi^* f_2$, which in turn implies $f_1 = f_2$. Hence πx is a smooth point of the unit ball of X/A.

For the proof of (ii) we assume that X is rotund. Let πx and πy be elements of a convex subset of the unit sphere of X/A such that x and y have best approximations a and b, respectively, in A. By the Hahn–Banach separation theorem there exists an f in the unit sphere of $(X/A)^*$ such that $f(\pi x) = f(\pi y) = 1$. Then $\pi^* f$ is in the unit sphere of $A^\perp$ and $\pi^* f(x - a) = \pi^* f(y - b) = 1$. This shows that $x - a$ and $y - b$ lie in a convex subset of the unit sphere of X. Since X is rotund, convex subsets of its unit sphere are singletons. Hence $x - a = y - b$, i.e., $\pi x = \pi y$ and this completes the proof.

COROLLARY. *Let X be a normed linear space and A a closed linear subspace of X. If X is smooth, resp. rotund, the quotient space X/A is smooth, resp. rotund, if and only if every element πx of the unit sphere of X/A with the property that x has no best approximation in A, is a smooth point, resp. an extreme point, of the unit ball of X/A.*

In the following theorem we characterize smoothness or rotundity of a normed linear space by that of certain of its quotient spaces. We remark that M. M. Day ([2]; p. 114) observed that smoothness, resp. rotundity, of all two-dimensional quotient spaces of a normed linear space is equivalent to rotundity, resp. smoothness, of its dual space, which in turn implies—but is not implied by—smoothness, resp. rotundity, of the space itself. The implication "(i) $\Rightarrow$ (ii)" of the theorem improves a result of V. Klee ([3]; Prop. 3.2) which states that the quotient spaces of a smooth or rotund normed linear space with respect to reflexive subspaces inherit these properties.

THEOREM. *Let X be a normed linear space. The following are equivalent:*

(i) *X is smooth, resp. rotund.*

(ii) *For every existence subspace A of X, the quotient space X/A is smooth, resp. rotund.*

(iii) *There exists a natural number $n \leqslant \dim X - 2$ such that for every n-dimensional subspace A of X, the quotient space X/A is smooth, resp. rotund.*

Proof. The implication "(i) $\Rightarrow$ (iii)" is an immediate consequence of the corollary and the implication "(ii) $\Rightarrow$ (iii)" is obvious.

For the proof of the smoothness part of the implication "(iii) $\Rightarrow$ (i)" we assume (iii) for some natural number $n \leqslant X-2$. If then X is not smooth, there exist distinct elements x_1^*, x_2^* of the unit sphere of X^* and an element x of the unit sphere of X such that $x_1^* x = x_2^* x = 1$. Now $x_1^{*-1}(\{0\}) \cap x_2^{*-1}(\{0\})$ contains an n-dimensional subspace A of X. The quotient space X/A is then not smooth because πx is in the unit sphere of X/A and $\pi^{*-1}(x_1^*)$, $\pi^{*-1}(x_2^*)$ are distinct elements of the unit sphere of $(X/A)^*$ satisfying $\pi^{*-1}(x_1^*)(\pi x) = \pi^{*-1}(x_2^*)(\pi x) = 1$, contradicting our assumption. (That πx is in the unit sphere of X/A follows from the well-known facts that $\|\pi x\| \leqslant \|x\| = 1$ and $\|\pi x\| = \sup\{x^* x : x^* \text{ in the unit sphere of } A^{\perp}\}$[3] together with the fact that, by definition of A, x_1^* and x_2^* are in the unit sphere of $A^{\perp}$. This last fact shows at the same time that $\pi^{*-1}(x_1^*)$ and $\pi^{*-1}(x_2^*)$ have the properties required.)

For the proof of the rotundity part of the implication "(iii) $\Rightarrow$ (i)" we assume again (iii) for some natural number $n \leqslant \dim X - 2$. If then X is not rotund, there exist distinct elements x_1, x_2 of the unit sphere of X and an element x^*

[3] This is a simple consequence of the Hahn–Banach separation theorem.

of the unit sphere of X^* such that $x^* x_1 = x^* x_2 = 1$. Now $x^{*-1}(\{0\})$ contains an n-dimensional subspace A of X whose intersection with the one-dimensional subspace of X spanned by $x_1 - x_2$ is zero. The quotient space X/A is then not rotund because πx_1 and πx_2 are distinct elements of a convex subset of the unit sphere of X/A, a contradiction to our assumption. (That πx_1 and πx_2 are distinct follows from the definition of A; that they are in the unit sphere of X/A follows as above; finally, that they lie in a convex subset of the unit sphere of X/A follows from $\pi^{*-1}(x^*)(\pi x_1) = \pi^{*-1}(x^*)(\pi x_2) = 1$.)

This completes the proof of the theorem.

Remark 1. Combining the implication "(i) $\Rightarrow$ (ii)" of the theorem with the result of V. Klee ([*3*]; Prop. 3.3) mentioned above, we obtain the following curiosity:

Let X be a separable normed linear space and A a closed linear subspace of X with codimension at least two. Then A is reflexive if and only if it is an existence subspace of X with respect to every smooth norm on X which is equivalent to the original one.[4]

We remark that the hypothesis of separability of X in this result cannot be dropped because (cf. M. M. Day [*1*]) there exist nonseparable Banach spaces that admit no equivalent smooth norm but contain nonreflexive existence subspaces of codimension at least two, e.g., for every infinite set I, the space $l_\infty(I)$ of all bounded real-valued functions on I with the supremum norm, where the subspace can be chosen to be the set of all elements of $l_\infty(I)$ that vanish at two distinct points of I.

I do not know whether in the above result "smooth" can be replaced by "simultaneously smooth and rotund"; cf., however, the remark of V. Klee ([*3*]; p. 62).

Remark 2. In order that a quotient of a smooth or rotund normed linear space with respect to a closed linear subspace inherits these properties, it is not necessary that the subspace be an existence subspace: If X is a nonreflexive Banach space whose dual X^* is rotund, resp. smooth, then X and all of its quotient spaces are smooth, resp. rotund (cf. M. M. Day [*2*]; pp. 112, 114) whereas X contains closed linear subspaces of codimension at least two that are not existence subspaces of X (cf. I. Singer [*4*]; p. 92).

In the following we give a different kind of example for the same fact:

Let I be a nonempty set, $(X_i : i \in I)$ a family of normed linear spaces and let p be a real number greater than one. $l_p(X_i : i \in I)$ denotes the linear space of

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[4] That a reflexive subspace of a normed linear space is an existence subspace follows by a simple compactness argument.

all elements x of the Cartesian product of the X_i with the property that the mapping $i \mapsto \|x(i)\|^p$ of I into the reals is summable, equipped with the norm $x = (\sum_{i\in I}\|x(i)\|^p)^{1/p}$ (cf. M. M. Day [2]; pp. 28–31 for details). If for every i in I, A_i is a closed linear subspace of X_i, $l_p(A_i : i \in I)$ is obviously a closed linear subspace of $l_p(X_i : i \in I)$ and it is easily verified that for every x in $l_p(X_i : i \in I)$

(i) $\text{dist}\,(x, l_p(A_i : i \in I)) = (\sum_{i\in I} \text{dist}\,(x(i), A_i)^p)^{1/p}$.

This shows that an x in $l_p(X_i : i \in I)$ has a best approximation a in $l_p(A_i : i \in I)$ if and only if for every i in I, $x(i)$ has the best approximation $a(i)$ in A_i, which implies in particular

(ii) $l_p(A_i : i \in I)$ is an existence subspace of $l_p(X_i : i \in I)$ if and only if for every i in I, A_i is an existence subspace of X_i.

Another consequence of (i) is

(iii) The mapping $x + l_p(A_i : i \in I) \mapsto (x(i) + A_i : i \in I)$ is an isometric isomorphism of $l_p(X_i : i \in I)/l_p(A_i : i \in I)$ onto $l_p(X_i/A_i : i \in I)$.

We shall show now how these observations can be used to construct the desired example. Let I have at least two elements. Choose for some i_0 in I a smooth, resp. rotund, normed linear space X_{i_0} whose dual is not rotund, resp. smooth and choose for every remaining i in I a normed linear space X_i with rotund, resp. smooth, dual. Since X_{i_0} is not reflexive, it contains a closed hyperplane A_{i_0} which is not an existence subspace of X_{i_0} (cf. I. Singer [2]; p. 92). For every i in I distinct from i_0 let A_i be a closed linear subspace of X_i, different from zero and X_i. Then by (ii), $l_p(A_i : i \in I)$ is not an existence subspace of $l_p(X_i : i \in I)$. However, by (iii) we have that $l_p(X_i : i \in I)/l_p(A_i : i \in I)$ is isometrically isomorphic to $l_p(X_i/A_i : i \in I)$ and the latter space is smooth, resp. rotund, because all the X_i/A_i are. Finally, the dual of $l_p(X_i : i \in I)$ is $l_{p/(p-1)}(X_i{}^* : i \in I)$ (cf. M. M. Day [2]; p. 31) and this space is not rotund, resp. smooth, because $X_{i_0}^*$ is not.

In view of the last example it seems unlikely to me that—apart from the not very illuminating corollary above—a "reasonably simple" complete description can be given of the extent to which smoothness or rotundity of a normed linear space is inherited by its quotient spaces.

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